B.Sc. part-I(H), paper-I Lecturered By Dr. H.K. Yadav.
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Q.1. A cyclic group G with a generator of finite order n, is isomorphic to the multiplicative group of n, nth roots of unity.

Proof: Since, we know that, if G be a cyclic group with a generator of finite order n, then G contains exactly n distinct elements namely

$$a, a^2, a^3, \ldots a^n = e$$

Now, let G' be the multiplicative group of n, nth roots of unity so that

$$G' = \{x : x \text{ is a solution of } x^n = 1\}$$
$$= \{x : x \text{ is a solution of } x^{1/n} = e^{2\pi i/n}\}$$
$$= \{e^{2\pi r i/n} ; r = 0, 1, 2, 3, \ldots (n-1)\}$$

Also, $G = \{a^r ; r = 1, 2, 3, \ldots n\} = \{a^r ; r = 0, 1, 2, 3, \ldots (n-1)\}$

For $a^n = e = a^0$, by definition of e.

Let us consider the map $f : G \to G'$ such that

$$f(a^r) = e^{2\pi r i/n} \;\forall\; a^r \in G.$$

Now, $f(a^r) = f(a^s) = e^{2\pi i r/n} = e^{2\pi s i/n} \Rightarrow r = s \Rightarrow a^r = a^s$

$\Rightarrow f$ is one-one

Further, $o(G) = n = o(G')$

$\Rightarrow f$ is onto

Now, $f(a^r \cdot a^s) = f(a^{r+s}) = e^{2\pi(r+s)i/n} = e^{2\pi r i/n} \cdot e^{2\pi s i/n}$
$= f(a^r) \cdot f(a^s)$

that is, $f(a^r \cdot a^s) = f(a^r) f(a^s) \;\forall\; a^r, a^s \in G$

$\Rightarrow f$ preserves composition in G and G'.

$\Rightarrow f$ is an isomorphism

$\Rightarrow$ Hence, $(G, *) \cong (G', *)$.

Q.2. A finite group of order n containing an element of order n must be cyclic.

Proof: Let G be a finite group of order n and let $a \in G$ such that $o(a) = n$. Then $o(G) = n = o(a)$.
We have to prove that G is cyclic.

Let H be a cyclic group generated by a, then
$o(H) = o(a) = n$ [$\because$ order of a cyclic group is equal to the order of its generator)

$\Rightarrow$ H can be expressed as
$$H = \{a^r ; r = 1,2,3,\ldots n\}.$$
Since, G is a group.
then, $\Rightarrow a \in G \Rightarrow a^r \in G$ for every integral value of r.
Thus, $H \subseteq G$
Moreover, $o(G) = n = o(H)$
$\therefore H = G$. but H is cyclic $\Rightarrow G$ is cyclic.

Q.3. Every subgroup of a cyclic group is cyclic.
Proof :. Let $G = \{a\}$ be a cyclic group generated by a. If $H = G$ or $\{e\}$, then obviously H is cyclic.
Now, Let H is a proper subgroup of G.
H contains the element of integral power of a.
If $a^s \in H \Rightarrow a^{-s} \in H$. Therefore, H contains elements which are positive as well as negative integral of a. Let K be the least positive integer such that $a^K \in H$. We have to show that $H = \{a^m\}$.
Now, Let $a^t \in H$. Then, by division algorithm
Then, exist of $q, r \in Z$ such that
$$t = Kq + r, \quad 0 \le r < K$$
Now, $a^K \in H \Rightarrow (a^K)^q \in H \Rightarrow a^{Kq} \in H \Rightarrow (a^{Kq})^{-1} \in H \Rightarrow a^{-Kq} \in H$.
Also, $a^t \in H$, $a^{-Kq} \in H \Rightarrow a^t a^{-Kq} \in H \Rightarrow a^{t-Kq} \in H \Rightarrow a^r \in H$
$\because K$ is the least positive integer such that $a^K \in H$ and $0 \le r < K$.
$\therefore r$ must be equal to 0
$\Rightarrow t = Kq$
$\therefore a^t = a^{Kq} = (a^K)^q$
$\Rightarrow$ every element $a^t \in H$ is of the form $(a^K)^q$
$\Rightarrow H$ is cyclic with generator a^m.